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Wilson-'t Hooft operators and the theta angle

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Abstract: We consider $(3+1)$-dimensional $\mathrm{SU}(N)/\mathbb{Z}_N$ Yang-Mills theory on a space-time with a compact spatial direction, and prove the following result: Under a continuous increase of the theta angle $\theta \to \theta + 2\pi$, a 't Hooft operator $T(\gamma)$ associated with a closed spatial curve γ that winds around the compact direction undergoes a monodromy $T(\gamma) \to T'(\gamma)$. The new 't Hooft operator $T'(\gamma)$ transforms under large gauge transformations in the same way as the product $T(\gamma)W(\gamma)$, where $W(\gamma)$ is the Wilson operator associated with the curve γ and the fundamental representation of $\mathrm{SU}(N)$.

Keywords: Differential and Algebraic Geometry, Gauge Symmetry, Duality in Gauge Field Theories.

Contents

1. Introduction

Wilson operators and 't Hooft operators constitute important observables in non-abelian Yang-Mills theory in $d = 3 + 1$ dimensions. In this paper, we will consider the gauge group

$$G \simeq \mathrm{SU}(N)/C\,, \tag{1.1}$$

where $C \simeq \mathbb{Z}_N$ denotes the center of $\mathrm{SU}(N)$. The basic Wilson operator $W(\gamma)$ associated with a closed spatial curve γ is then defined as

$$W(\gamma) = \frac{1}{N}\mathrm{Tr}\left(P\exp\int_\gamma A\right), \tag{1.2}$$

where A is the connection one-form, P denotes path ordering along γ, and Tr is the trace in the fundamental representation of $\mathrm{SU}(N)$. The operator $W(\gamma)$ is invariant under gauge transformations that can be continuosly deformed to the identity transformation. Under a general gauge transformation, $W(\gamma)$ is multiplied by an N-th root of unity determined by the class in $\pi_1(G) \simeq \mathbb{Z}_N$ of (the restriction to γ of) the gauge transformation.

The definition of the corresponding 't Hooft operator $T(\gamma)$ is less explicit [1]: On the complement of γ in space, $T(\gamma)$ is given by a G valued gauge transformation, whose restriction to another closed curve that links γ once represents the image of the generator 1 of $\mathbb{Z}_N$ under the isomorphism $\mathbb{Z}_N \simeq \pi_1(G)$. Such a transformation has a well-defined action on all the fields of the theory, but is obviously singular at the locus of γ. By deforming the transformation over a tubular neighbourhood of γ, we regularize it to a smooth transformation defined over all of space-time. The precise form of the regularization is of no consequence for the arguments of the present paper; the important point is that the resulting field configuration is smooth everywhere. The regularized transformation is however not a gauge transformation, so the 't Hooft operator $T(\gamma)$ thus defined has a non-trivial action also on gauge invariant states.

This definition of the 't Hooft operator in terms of a singular gauge transformation is ambigious in the sense that it allows for the multiplication of $T(\gamma)$ by a phase-factor, which may be an arbitrary gauge-invariant functional of the fields of the theory. It may even be impossible to give a globally valid prescription for fixing this ambiguity. Indeed, it has been stated in several papers (see for example [2][3][4]), that under a smooth increase $\theta \to \theta + 2\pi$ of the theta angle, $T(\gamma)$ undergoes a monodromy

$$T(\gamma) \to T'(\gamma) , \tag{1.3}$$

where the new 't Hooft operator $T'(\gamma)$ behaves as the product $T(\gamma)W(\gamma)$; it could be called a Wilson-'t Hooft operator. (On the other hand, the explicit expression (1.2) shows that the corresponding monodromy of the Wilson operator $W(\gamma)$ is trivial; $W(\gamma) \to W(\gamma)$.) This monodromy of operators associated with closed spatial curves is analogous to the Witten effect [5], which amounts to an increase of the electric charge of a magnetically charged dyonic particle state as $\theta \to \theta + 2\pi$ continuously.

However, we are not aware of any published proof of the monodromy transformation (1.3). The aim of the present paper is to provide such a proof, based on a topological obstruction that prevents a global definition of $T(\gamma)$. The obstruction will only be present if the curve γ represents a non-trivial homotopy class. We will therefore consider the theory on a spatial three-manifold X of the form

$$X \simeq S^1 \times \mathbb{R}^2 , \tag{1.4}$$

and let γ wind once around the S^1 factor of X. However, even if there is no topological obstruction against a global definition of the 't Hooft operator if we take the spatial manifold as $\mathbb{R}^3$, it is certainly natural to assume a similar monodromy transformation also in this case.

In the next section, we will review the interpretation of the 't Hooft operator $T(\gamma)$ in terms of the topology of principal G bundles over space. In section three, we will consider the topology of the group of gauge transformations. A gauge transformation may be winded in two different ways: along the curve γ, or over three-space as a whole. As discussed above, gauge transformations that are winded along γ have a non-trivial action on the Wilson operator. The transformation properties under gauge transformations that are winded over three-space as a whole are described by the theta angle. In section four, we will show how the interplay of these effects leads to the monodromy property (1.3).

2. 't Hooft operators

We begin by reviewing the classification of principal $G \simeq \mathrm{SU}(N)/C$ bundles over a low-dimensional compact connected space B. This follows from the first few homotopy groups of G:

$$\pi_i(G) \simeq \begin{cases} 0, & i = 0 \\ \mathbb{Z}_N, & i = 1 \\ 0, & i = 2 \\ \mathbb{Z}, & i = 3. \end{cases} \tag{2.1}$$

Thus, for a one-dimensional base space B, all G bundles are trivial. For a two- or three-dimensional B, they are classified by a characteristic class

$$w_2 \in H^2(B, \mathbb{Z}_N)\,, \tag{2.2}$$

known as the second Stiefel-Whitney class in mathematics or the discrete magnetic flux in physics. For a four-dimensional B there is an additional characteristic class

$$c_2 \in H^4(B, \mathbb{Q})\,, \tag{2.3}$$

known as the second Chern class or the instanton number. It is related to the second Stiefel-Whitney class w_2 as

$$c_2 = \frac{1}{2}\left(\frac{1}{N} - 1\right) \bar{w}_2 \cup \bar{w}_2 \bmod H^4(B, \mathbb{Z})\,, \tag{2.4}$$

where $\bar{w}_2 \in H^2(B, \mathbb{Z})$ denotes an arbitrary lifting of w_2 to an integral class. (An instructive proof of this relation can be found in e.g. [6].) In higher dimensions, there are further invariants, but they will not be needed in the present paper.

Consider now a state of finite energy in Yang-Mills theory with gauge group G on the spatial manifold $X \simeq S^1 \times \mathbb{R}^2$. As we go to infinity in the $\mathbb{R}^2$ factor, all physical data must approach their vacuum values. We may therefore add the points at infinity, thereby replacing X by the compact space

$$X' \simeq S^1 \times S^2. \tag{2.5}$$

However, while a G bundle P over X is necessarily trivial (since $H^2(X, \mathbb{Z}_N) \simeq 0$), this is not so for a G bundle P' over X'; according to the previous paragraph, such bundles are classified by a characteristic class $w_2' \in H^2(X', \mathbb{Z}_N) \simeq \mathbb{Z}_N$.

It is now easy to understand the action of an 't Hooft operator $T(\gamma)$ associated with a closed curve γ that winds once around the S^1 factor of X or X': When acting on a state $|\psi\rangle$ with a definite value w_2' of the second Stiefel-Whitney class, it produces another state $|\tilde{\psi} >= T(\gamma)\,|\psi\rangle$ for which the second Stiefel-Whitney class takes the value $\tilde{w}_2'$ given by

$$\tilde{w}_2' = w_2' + 1. \tag{2.6}$$

(In this formula, we identify a class in $H^2(X', \mathbb{Z}_N)$ with its image under the isomorphism $H^2(X', \mathbb{Z}_N) \simeq \mathbb{Z}_N$.)

3. Large gauge transformations

This section is largely inspired by [7].

Let P' be a G bundle over $X' \simeq S^1 \times S^2$, characterized by its value $w_2' \in H^2(X', \mathbb{Z}_N)$ of the second Stiefel-Whitney class, as described in the previous section. We let $\mathcal{G}$ denote the group of gauge transformations, i.e. the group of bundle automorphisms of P'. It is not connected; the component of $\mathcal{G}$ containing the identity transformation is a normal subgroup, which we denote as $\mathcal{G}_0$. Physical states must be invariant under $\mathcal{G}_0$, but they need not be invariant under all of $\mathcal{G}$. Their transformation properties may be given by an arbitrary character of the quotient group of homotopy classes of gauge transformations

$$\Lambda \simeq \mathcal{G}/\mathcal{G}_0. \tag{3.1}$$

We will need to understand the structure of the discrete abelian group Λ. Let $\Lambda_\gamma \simeq \pi_1(G) \simeq \mathbb{Z}_N$ be the group of homotopy classes of gauge transformations for the trivial bundle over the spatial curve γ that is obtained by restricting the bundle P' to γ. Let $\Lambda_0 \simeq \pi_3(G) \simeq \mathbb{Z}$ be the subgroup of Λ consisting of homotopy classes of gauge transformations of P' that are trivial when restricted to γ. We thus have a short exact sequence

$$0 \to \Lambda_0 \xrightarrow{i} \Lambda \xrightarrow{r} \Lambda_\gamma \to 0\,, \tag{3.2}$$

where i and r are the obvious inclusion and restriction maps respectively. In other words, the group Λ is an extension of $\Lambda_\gamma \simeq \mathbb{Z}_N$ by $\Lambda_0 \simeq \mathbb{Z}$. To describe this extension precisely, we choose a $\lambda \in \Lambda$ such that $r(\lambda)$ equals the image of the generator 1 of $\mathbb{Z}_N$ under the isomorphism $\mathbb{Z}_N \simeq \Lambda_\gamma$. Since $\lambda^N \in \ker r$ and the sequence is exact, $\lambda^N \in \operatorname{Im} i$, so

$$\lambda^N = \Omega^k\,, \tag{3.3}$$

where Ω is the generator of $\Lambda_0 \simeq \mathbb{Z}$ and k is some integer, which depends on the choice of λ. (Here we have switched to a multiplicative rather than additive notation for the group operations.)

To compute the integer k, we consider the four-dimensional space

$$Y \simeq S^1 \times X' \simeq S^1 \times S^1 \times S^2. \tag{3.4}$$

(As will become clear, this auxiliary space should not be thought of as a space-time.) We construct two G bundles P^λ and P^Ω over Y by first extending the given bundle P over the cylinder $I \times X'$, where I is an interval, and then gluing the ends together with gluing data λ or Ω respectively. We then have that

$$N c_2^\lambda = k c_2^\Omega\,, \tag{3.5}$$

where c_2^λ and c_2^Ω are the second Chern classes of the bundles P^λ and P^Ω respectively. The class $c_2^\Omega \in H^4(Y, \mathbb{Q})$ is given by the image of the element 1 of $\mathbb{Q}$ under the isomorphism $\mathbb{Q} \simeq H^4(Y, \mathbb{Q})$. According to (2.4), the class $c_2^\lambda \in H^4(Y, \mathbb{Q})$ is determined modulo $H^4(Y, \mathbb{Z})$ by the second Stiefel-Whitney class $w_2^\lambda \in H^2(Y, \mathbb{Z}_N)$ of P^λ. The latter class is determined by its restrictions to the factors $S^1 \times S^1$ and S^2 on the right hand side of (3.4). The restriction of w_2^λ to $S^1 \times S^1$ is in fact given by $r(\lambda) \in H^2(S^1 \times S^1, \mathbb{Z}_N) \simeq \Lambda_\gamma$. The restriction of w_2^λ to S^2 equals the second Stiefel-Whitney class $w_2' \in H^2(S^2, \mathbb{Z}_N) \simeq H^2(X', \mathbb{Z}_N)$ of P'. Thus

$$w_2^\lambda = p_1^*(r(\lambda)) + p_2^*(w_2')\,, \tag{3.6}$$

where p_1 and p_2 are the projections from Y to $S^1 \times S^1$ and S^2 respectively. A small calculation now gives

$$c_2^\lambda = \frac{1}{N} p_1^*(r(\lambda)) \cup p_2^*(w_2') \bmod H^4(Y, \mathbb{Z}). \tag{3.7}$$

Putting everything together, we find that $k = w_2' \bmod N$, where w_2' denotes the image of the second Stiefel-Whitney class of P' under the isomorphism $H^2(X', \mathbb{Z}_N) \simeq \mathbb{Z}_N$. (Since λ is only defined modulo Ω, we can only determine k modulo N.)

In summary, we have found that the group Λ is generated by the elements λ and Ω, subject to the relation

$$\lambda^N = \Omega^{w_2' \bmod N}. \tag{3.8}$$

4. The monodromy

A physical state $|\psi\rangle$ is characterized by a certain value w_2' of the second Stiefel-Whitney class, as described in section two. Its transformation properties under the discrete abelian group Λ described in the previous section can be specified by the eigenvalues $e^{i\theta}$ and $e^{i\phi}$ of the generators Ω and λ respectively:

$$\begin{aligned} \Omega\,|\psi\rangle &= e^{i\theta}\,|\psi\rangle \\ \lambda\,|\psi\rangle &= e^{i\phi}\,|\psi\rangle\,. \end{aligned} \tag{4.1}$$

As the notation suggests, θ is indeed the theta angle parameter of Yang-Mills theory. The relation (3.8) implies that

$$e^{i\phi N} = e^{i(w_2'+nN)\theta} \tag{4.2}$$

for some integer n. If we follow a particular solution to this equation under a continuous increase $\theta \to \theta + 2\pi$, the eigenvalue $e^{i\phi}$ undergoes the monodromy

$$e^{i\phi} \to e^{i\phi} e^{2\pi i w_2'/N}. \tag{4.3}$$

Acting with an 't Hooft operator $T(\gamma)$ on $|\psi\rangle$ produces another state $|\tilde{\psi}>= T(\gamma)\,|\psi\rangle$ with the value $\tilde{w}_2' = w_2' + 1$ of the second Stiefel-Whitney class. Repeating the above argument, we find that the corresponding eigenvalue $e^{i\tilde{\phi}}$ of the generator λ undergoes the monodromy

$$e^{i\tilde{\phi}} \to e^{i\tilde{\phi}} e^{2\pi i (w_2'+1)/N}. \tag{4.4}$$

The different monodromy properties of the two states mean that the 't Hooft operator must undergo a monodromy

$$T(\gamma) \to T'(\gamma). \tag{4.5}$$

The quotient $\hat{W}(\gamma) = T'(\gamma)T^{-1}(\gamma)$ can be characterized by its transformation property under λ:

$$\lambda \hat{W}(\gamma) \lambda^{-1} = e^{2\pi i/N} \hat{W}(\gamma). \tag{4.6}$$

But this agrees with the transformation property of the Wilson operator $W(\gamma)$ in the fundamental representation of SU(N) as defined in (1.2). So although the present arguments do not give an exact description of $T'(\gamma)$ (which would depend on the precise prescription for regularizing the 't Hooft operators in the vicinity of γ), we can conclude that $T'(\gamma)$ indeed transforms in the same way as the product $T(\gamma)W(\gamma)$ under gauge transformations.

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